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Arithmetical proofs in Arabic algebra

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1. Introduction

Much attention has been paid by historians of Arabic mathematics to the proofs by geometry of the rules for solving quadratic equations. The earliest Arabic books on algebra give geometric proofs, and many later algebraists introduced innovations and variations on them. The most cited authors in this story are al-Khwārizmī, Ibn Turk, Abū Kāmil, Thābit ibn Qurra, al-Karajī, al-Samawʾal, al-Khayyām, and Sharaf al-Dīn al-Ṭūsī.[2] What we lack in the literature are discussions, or even an acknowledgement, of the shift in some authors beginning in the eleventh century to give these rules some kind of foundation in arithmetic. Al-Karajī is the earliest known algebraist to move away from geometric proof, and later we see arithmetical arguments justifying the rules for solving equations in Ibn al-Yāsamīn, Ibn al-Bannāʾ, Ibn al-Hāʾim, and al-Fārisī. In this article I review the arithmetical proofs of these five authors. There were certainly other algebraists who took a numerical approach to proving the rules of algebra, and hopefully this article will motivate others to add to the discussion.

To remind readers, the powers of the unknown in Arabic algebra were given individual names. The first degree unknown, akin to our x, was called a *shayʾ* (thing) or *jidhr* (root), the second degree unknown (like our x^2) was called a *māl* (sum of money),[3] and the third degree unknown (like our x^3) was named a *kaʿb* (cube). Higher powers were usually named in terms of *māl* and *kaʿb*, like *māl kaʿb* for the sixth power. Simple numbers were often counted in dirhams, a denomination of silver coin. Only positive numbers were acknowledged in Arabic mathematics, and the solutions to simplified equations were calculated from the coefficients. So instead of a single standard form $ax^2 + bx + c = 0$ for quadratic equations, Arabic algebraists classified six types. Three are simple, with two terms, and three are composite, with three terms. We give them here in the order from al-Khwārizmī's early ninth-century *Book of algebra* (*Kitāb al-jabr wa'l-muqābala*):[4]

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[1] I thank Mahdi Abdeljaouad for his comments on an earlier version of this article, and for his help in parsing al-Karajī's remarks from *ʿIlal ḥisāb*.

[2] Studies that examine the geometric proofs of at least three of these authors are [Djebbar 1981; Dold-Samplonius 1987; Sesiano 1999].

[3] Because there is no good English translation of *māl*, I leave it untranslated. Also, I write its plural with the English suffix: *māl*s.

[4] [al-Khwārizmī 2009, 97ff]. Some later authors rearranged the first three equations, but the order of the composite equations remained the same in all the authors I quote.

Simple equations
Type 1. *māl*s equal roots (like our $ax^2 = bx$)
Type 2. *māl*s equal number ($ax^2 = c$)
Type 3. roots equal number ($bx = c$)

Composite equations
Type 4. *māl*s and roots equal number (like our $ax^2 + bx = c$)
Type 5. *māl*s and number equal roots ($ax^2 + c = bx$)
Type 6. roots and number equal *māl*s ($bx + c = ax^2$)

Solutions to the simple equations are easy, but the composite equations each require a special rule. What I call the "standard rule" in Arabic algebra begins by normalizing the equation, that is, by setting the number of *māl*s to 1. If the number is greater than 1 then one "returns" (*radda*) the *māl*s to one *māl*, and if there are fewer than one *māl* then one "completes" (*ikmāl*) the fraction of a *māl* to one *māl*. Al-Karajī describes what follows this step for the type 4 equation in his *al-Fakhrī*: "So once the *māl* became one *māl*…you halved the things [i.e. take half of b], and you multiplied the number of its half it by itself. Then you added the outcome to the number [c], and you took the root of the result of that, and you subtracted from it half the roots. What remained is the root of the *māl*".[5] To contort this into a modern formula, the solution to $x^2 + bx = c$ is $x = \sqrt{(\frac{1}{2}b)^2 + c} - \frac{1}{2}b$. Keep in mind that the rule was read as a sequence of operations to be performed on known numbers, which is not the same thing as the static modern formula shown above. As long as one is aware of the differences, modern formulas can help clarify for us the way the medieval rhetorical descriptions proceed, so we will use them in this article. The standard Arabic rules for equations of types 5 and 6 are similar. In modern notation the type 5 equation $x^2 + c = bx$ is solved by $x = \frac{1}{2}b \pm \sqrt{(\frac{1}{2}b)^2 - c}$, and the type 6 equation $bx + c = x^2$ is solved by $x = \sqrt{(\frac{1}{2}b)^2 + c} + \frac{1}{2}b$.

Algebraic equations in medieval Arabic are always stated with conjugations of the verb *ʿadala*, meaning "to equal". This word is rarely used in mathematics outside the context of algebra. Instead, other words were used to equate numbers, lines, angles, etc.: *mithl*, conjugations of *sawiya*, the prefix *ka-*, and the implied verb "to be".[6] It is thus easy to see if a particular equating is intended to be an algebraic equation. This is important for understanding whether an author's line of reasoning takes place in the realm of arithmetic, or whether the setting is algebra itself. To make the distinction clear in the translations, from now on I write "*equal*" in italics when it comes from *ʿadala* and indicates an algebraic equation. Also, when rendering the rhetorical mathematics in modern notation I use an arrow (→) to indicate the result of an operation, and the equal sign (=) for algebraic equations.

[5] [Saidan 1986, 149.9].
[6] [Oaks 2010].

The books of the five authors surveyed in this article are:

al-Karajī (fl. Late 10th to early 11th c.)

al-Fakhrī fī ṣināʿat al-jabr wa l-muqābala (*[Book] of al-Fakhrī on the art of algebra*). Completed ca. 401H/1010-1 CE.[7]

ʿIlal ḥisāb al-jabr wa'l-muqābala wa'l-burhān ʿalayhi (*Causes of calculation in algebra and their proof[s]*[8]). Written after *al-Fakhrī*.[9]

al-Kāfī fī l-ḥisāb (*Sufficient [book] on calculation*). Written after *al-Fakhrī*.[10]

Ibn al-Yāsamīn (d. 1204)

Talqīḥ al-afkār fī'l-ʿilm bi-rushūm al-ghubār (*Grafting of opinions of the work on dust figures*)[11]

Ibn al-Bannāʾ (1256-1321),

Kitāb al-uṣūl wa'l-muqaddimāt fī'l-jabr wa'l-muqābala (*Book on the fundamentals and preliminaries in algebra*). Late 13th century.[12]

Rafʿ al-ḥijāb ʿan wujūh aʿmāl al-ḥisāb (*Lifting the veil from the face of the operations of arithmetic*). 1301.[13]

Ibn al-Hāʾim (1352-1412)

Sharh al-urjūza al-Yāsmīnīyya (*Commentary on the poem of al-Yāsamīn*). 1387.[14]

al-Fārisī (d. ca. 1320)

Foundation of rules on elements of benefits (*Asās al-qawāʿid fī uṣūl al-fawāʾid*). Late 13th century.[15]

I use the English word "proof" rather loosely in this article to mean any kind of explanation as to why a rule is valid. There is no single word meaning "proof" in a strict sense in the books listed above. The common word for "proof" in the Arabic mathematics connected with Greek geometry is *burhān*, pl. *barāhīn*. This is the word used in the Ishaq-Thābit translation of the *Elements*, and it appears in the geometrical works of the Banū Mūsā, Thābit ibn Qurra, Ibn al-Haytham, and others. In al-Khwārizmī's and Abū Kāmil's proofs for the rules to solve equations we find instead the word *ʿilla*, pl. *ʿilal*, which more literally means "cause". Later in his book Abū Kāmil uses *burhān* for proofs of basic rules for multiplying algebraic monomials and

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7 Edited in [Saidan 1986, 95-308].

8 The word "proof" is singular in the Arabic, but the plural fits the meaning better in English. The same singular form occurs within the book and in other books as well.

9 Edited in [Saidan 1986, 353-369].

10 Edited in [al-Karajī 1986].

11 Edited in [Ibn al-Yāsamīn 1993].

12 Edited in [Saidan 1986, 505-585].

13 Edited in [Ibn al-Bannāʾ 1994].

14 Edited with partial French translation in [Ibn al-Hāʾim 2003].

15 Edited in [al-Fārisī 1994].

polynomials, and there is no discernible difference between the technique or wording in these proofs and the ones using *ʿilla*. Also, he uses a third word, *bayān*, for other proofs. A look through al-Fārisī's book shows a similarly cavalier attitude toward word choice. Both *bayān* and *burhān* appear numerous times in this book, and also less frequently *ʿilla* and *dalīl*. But some authors, including al-Karajī, seem to have distinguished between *burhān* and *ʿilla*. Whatever differences individual authors observed in these words remain for now a topic for another study.

The arithmetical proofs of our five authors rest on one of three kinds of rule, and these rules themselves are usually not proven. One kind is the rule for multiplying binomials which in modern notation is $(a+b)^2 \to a^2 + b^2 + 2ab$, or a modification that we today call "completing the square". Another kind is arithmetical restatements of Propositions II.5 and II.6 of Euclid's *Elements*, though Euclid is never cited explicitly. Proposition II.6 is used for type 4 and 6 equations, and Proposition II.5 for type 5 equations. And in one set of proofs Ibn al-Bannāʾ appropriates a rule from finger reckoning for multiplying numbers as the foundation for the proofs.

2. An overview of geometric proofs before al-Karajī

Four main extant works written before the time of al-Karajī give proofs to the rules for solving composite equations.[16] They are

al-Khwārizmī (early 9th c.)
Kitāb al-jabr wa'l-muqābala (*Book of algebra*)[17]
Ibn Turk (early 9th c.)
Kitāb al-jabr wa'l-muqābala (*Book of algebra*)[18]
Thābit ibn Qurra (late 9th c.)
Qawl fī taṣḥīḥ masāʾil al-jabr bi'l-barāhīn al-handasiyya
(*Establishing the correctness of algebra problems by geometric proofs*)[19]
Abū Kāmil (late 9th c.)
Kitāb fī'l-jabr wa'l-muqābala (*Book on algebra*)[20]

[16] I know of only two other works written prior to al-Karajī's *al-Fakhrī* that give geometric proofs. One is the *Collection of geometrical problems* of Nuʿaim ibn Muḥammad ibn Mūsā (late 9th c.). The fourth of the 42 problems in this book gives three proofs each for the solutions to equations of the types $x^2 + a = bx$ and $x^2 + b = ax$. Nuʿaim likely adapted his proofs from Thābit's treatise. Also, Jacques Sesiano mentions an unpublished anonymous text from 395H (1004-5 CE) in which the diagrams are constructed [Sesiano 1999, 83].
[17] Edited with French translation in [al-Khwārizmī 2007], and with English translation in [al-Khwārizmī 2009].
[18] Edited with English and Turkish translations in [Sayili 1962]
[19] Edited with German translation in [Luckey 1941] and with French translation in [Rashed 2009].
[20] Facsimile of the Istanbul MS is published in [Abū Kāmil 1986]. Edited with German translation in [Abū Kāmil 2004], and with French translation in [Abū Kāmil 2012].

All proofs in these books for the rules for solving equations are based in geometry, with different authors giving variations not found in the others. The texts of al-Khwārizmī and Ibn Turk were written in Baghdad during the reign of al-Maʾmūn (813-833 CE). Both prove the rules in the context of specific equations. For example, al-Khwārizmī works through the type 4 equation in terms of the example "a *māl* and ten roots *equal* thirty-nine dirhams" ($x^2 + 10x = 39$), and Ibn Turk's is the same but with 24 in place of 39.[21] Both authors represent the thing by a line and the *māl* by a square on that line, and the arguments unfold by comparing equal lines and areas. No recourse is made to Euclid's *Elements*. In fact, the only evidence of Greek influence is that the vertices in the diagrams are labeled with letters. Jens Høyrup has convincingly argued that these proofs were adapted from techniques of practical geometers. The proofs, then, were not presented as an application of Greek-style mathematics to practical Arabic algebra. Rather, they reflect a common feature of practical mathematics of the time.[22]

Thābit ibn Qurra took an entirely different approach in his short work. He framed his proofs somewhat in the style of Euclid's *Data*, tracking as he goes which numbers are known or given. And instead of the cut-and-paste approach of al-Khwārizmī and Ibn Turk, he grounds his proofs in Propositions II.5 and II.6 of Euclid's *Elements*. This reorientation toward Greek geometry, coupled with the fact that the treatise does not give any introductory definitions or any worked-out problems, indicate that Thābit's proofs were intended for mathematicians and not for practitioners.

Thābit diverged from Euclid as well as other Arabic algebraists in the way he organized his proofs. Rather than state the rule for solving an equation and then proving it, he states only the equation itself and then he effectively derives the rule in the course of analyzing his diagram. For example, here is how he begins the type 4 proof:

> The first principle is: a *māl* and roots *equal* a number [$x^2 + bx = c$]. The way to solve (*istikhrāj*) this, by proposition six of the second book of the work of Euclid, is as I describe. Let the *māl* be square ABGD…[23]

This way the steps of the solution are justified as they are presented. Later we will see that in *al-Fakhrī* al-Karajī also gives derivations, this time arithmetical, for the composite equations by a technique borrowed from Diophantus.

Abū Kāmil's main influences were al-Khwārizmī and Euclid. In addition to the standard rules to find the "thing" for each composite equation, he added rules to find the *māl* directly. He also greatly expanded the range of propositions that receive proofs. He gave a total of 39 proofs by geometry to various propositions in algebra and arithmetic. In addition, ten propositions receive arithmetical proofs in the style of propositions from Books VII to IX of Euclid's *Elements*, and a couple propositions are proven by algebra. All proofs for the rules to solve equations, 15

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[21] [al-Khwārizmī 2007, 109.1; Sayili 1962, 145.17].
[22] See [Høyrup 1986; Oaks 2012].
[23] Translation adapted from [al-Khwārizmī 2009, 34].

in all, are by geometry (some rules are given more than one proof).[24] Unlike Thābit, Abū Kāmil's propositions follow the standard format of (a) statement of the rule, followed by (b) the proof(s).

We should note that some algebraists, like ʿAlī al-Sulamī (10th c.) and Ibn Badr (13th c.?), present the rules in their books on algebra with no proofs at all. Shorter arithmetic books treating algebra will also frequently omit proofs, like Ibn al-Bannāʾ's *Talkhīṣ aʿmāl al-ḥisāb* (*Condensed [book] on the operations of arithmetic*) (ca. 1300).

For the rest of this paper I review the arithmetical proofs given in the books of our five algebraists, after which I offer some concluding remarks on proof in Arabic algebra at the end. This is followed by an appendix listing the types of proofs in each book.

3. al-Karajī's *al-Fakhrī*: Geometric proofs and arithmetical derivations

Al-Karajī wrote his mathematical works in Baghdad, and he probably completed his algebra book *al-Fakhrī* in 401H (1010/1 CE). His main influences were Diophantus of Alexandria's *Arithmetica*, written in Greek sometime in late antiquity, and Abū Kāmil's *Book on algebra*. Al-Karajī borrowed from both books not only for many of his 255 worked-out problems and for material in the preliminary chapters, but also for the overall organization of the work.

Diophantus and Abū Kāmil both left their marks in the rules for solving equations. Al-Karajī first gives the standard Arabic rule for finding the "thing", whose first step is the normalization of the equation. Then he gives Diophantus's rule for finding the "thing" without first dividing by the number of *māl*s, and third, he gives Abū Kāmil's rule for finding the *māl* directly.[25] After presenting the three rules al-Karajī gives geometric proofs for each of them, and he concludes with an arithmetical derivation of the standard rule "in the manner followed by Diophantus".

One way al-Karajī deviated from Abū Kāmil is that he sharply scaled back the number of proofs. He writes just before his first proof for the type 4 equation:

> I made a determination in this book to strip it of proofs (*barāhīn*), lengthy explanations, and numerous examples. But I cannot avoid giving a brief summary of the proof[s] (*burhān*) for the connected problems[26] and the cause (*ʿilla*) of halving the roots and what is associated with it.[27]

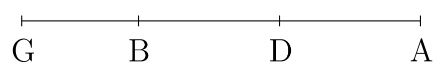
The only proofs he gives, in fact, are to the rules for solving the composite equations. Like the proofs of Thābit and Abū Kāmil, these proofs are grounded in *Elements* II.5 and II.6. But unlike those authors, his diagrams for the proofs to the standard Arabic rule consist of only a single line. This may be because his proofs to Diophantus's and Abū Kāmil's rules require that he add a

[24] See [Oaks 2011] for an analysis of the proofs.
[25] See [Oaks, forthcoming].
[26] I.e., the composite equations.
[27] [Saidan 1986, 151.8]. The "what is associated with it" is the rest of the rule, after taking half the roots.

dimension to the diagram. Whatever the reason, his proofs to the standard rule have a more arithmetical feel to them than the proofs of earlier algebraists, and this may have facilitated their conversion to an arithmetical foundation later in Ibn al-Yāsamīn and Ibn al-Hāʾim.

Here is al-Karajī's proof for the standard rule for the type 4 equation, worked out in the context of the equation $x^2 + 10x = 39$. (He had addressed the normalization of the equation just before this):[28]

> So for this, a *māl* and ten roots *equal* thirty-nine units. The way to find the root is that you multiply half the roots by itself, and you add it to the number, and you take the root of the outcome, and you subtract from it half the roots.
>
> Proof (*burhān*) of this is that we make line BG a thing, and line AB ten in number, and we divide it into two halves at point D. We want to extend [it by] line BG. You knew that if any line is appended with an extension, then multiplying the line with the extension by the extension, and adding to it the square of half the line, gives the square on all of the half line with the extension. So in this problem line AB is extended by line BG, so the product of all of line AG by line BG, and line DB by itself, is equal to line DG by itself, as Euclid demonstrated in his work.[29] And we knew that line AB is ten, and line BG is a root of the *māl*, and if you multiplied all of line AG by line BG it gave thirty-nine units, which is equal to the *māl* with ten of its roots. So if you added to that line DB, which is five, by itself, it gave sixty-four, and a root of that is line DG. So line DG is known, which is eight, and line DB is five, leaving line BG three, which is the root of the *māl*, and the *māl* is nine. And this is its figure:



After proving the rules of Diophantus and Abū Kāmil using two-dimensional diagrams, al-Karajī gives an arithmetical solution/proof to the first rule. Like Thābit's proofs, it takes the form of a derivation:

> And if you wanted to find the root of the *māl* in the manner followed by (*maḏhab*) Diophantus, you searched for a number which, if added to a *māl* and ten things [$x^2 + 10x$], has a root. It is nothing but twenty-five, which added to a *māl* and ten things has a root that is a thing and five dirhams [$x + 5$]. And you knew that a *māl* and ten things are thirty-nine units, so if you removed the *māl* and ten things, and you put in its place thirty-nine units, they became sixty-four units. So its root is eight, and that *equals* a thing and five dirhams. So the thing *equals* three dirhams, which is the root of the *māl*.[30]

Here al-Karajī completes the square for "a *māl* and ten things" to find, in modern notation, that

$$(x + 5)^2 \to x^2 + 10x + 25.$$

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[28] [Saidan 1986, 151.11].
[29] *Elements* II.6.
[30] [Saidan 1986, 154.13].

(Keep in mind that the only operation intended here is the squaring of the "thing and five dirhams". The plus signs indicate aggregation only, and the x^2 is a single term, *māl*.) Substituting 39 for the $x^2 + 10x$ gives

$$(x + 5)^2 \to 39 + 25 \to 64.$$

Thus $x + 5 = 8$, stated as an equation, so $x = 3$. In this derivation the work takes place in the context of the operation of squaring the $x + 5$, and the equation is used to make a substitution. Although he reformulates an equation at the end, no equation is reformulated at the end for the type 5 derivation.

The sample equation for type 5 is $x^2 + 21 = 10x$. Squaring $x - 5$ or $5 - x$ gives $x^2 + 25 - 10x$. Replacing the $10x$ with $x^2 + 21$ gives the result of the squaring as 4, and the two solutions follow. Two parts are missing for al-Karajī's treatment of the type 6 equation: the statement and examples for Diophantus's rule (though the proof is included) and the derivation by the method of Diophantus. These were most likely originally given by al-Karajī and were later omitted in the manuscripts.[31]

4. al-Karajī's *ʿIlal ḥisāb*

Al-Karajī wrote his short work *ʿIlal ḥisāb* (*Causes of calculation*) after *al-Fakhrī*. He begins it by describing the difficulty that those "seeking knowledge of calculation", i.e., arithmeticians, had with his geometric proofs:

> I aspired to establish the proof[s] (*burhān*) for what was prescribed in halving the roots, etc., establishing the proof[s] (*burhān*) by means of lines and figures. The knowledge [coming] from this is clear from evidence that cannot be refuted, and does not rely on any other [knowledge]. Then I saw that people seeking knowledge of calculation found it difficult to understand the correctness [of the rules] by means of those lines and figures, given their understanding by means of the tongue and hand.[32] I discovered that many people found it very difficult when reading them [i.e. the proofs] in books.
>
> And I decided because of this to make the proofs (*barāhīn*) in this book easier, because the foundation [1] that eliminates the uncertainty of [both] everything that was drawn and the arguments that make up each cause (*ʿilla*), and [2] that is independent of those drawings, is instead proofs (*barāhīn*) by arithmetic, using algebra and number, [which] puts an end to the incomprehensibility of the realm of lines and figures. Instead of having to piece together the

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[31] See [Oaks forthcoming].

[32] Geometry was viewed as a visual science, while arithmetic was traditionally regarded as a spoken science. This is partly due to the prevalence of finger-reckoning, in which intermediate results of calculations are stored by positioning the fingers in particular ways. This method of calculation involved the tongue (for speaking) and the hand (for storing numbers).

> validity of the proof[s] (*burhān*) from what you drew, everything for learning the science of arithmetic can be seen in this book.[33]

Now, instead of geometric proofs for the three equations he reworks the "manner followed by Diophantus" into arithmetical proofs to the standard rules for types 4 through 6. What is especially fortunate is that we can see the argument for the type 6 equation that is missing in *al-Fakhrī*. Here is the proof for type 4, to compare with the derivation just translated from *al-Fakhrī*:

> If you wanted to show the reason for saying, in the first problem, which is the *māl* and roots *equal* a number, that the way [to solve] it is that you halve the roots, and you multiply [it] by itself, and add to the total the number *equal* to the *māl* and roots, then take the root of the sum, then subtract from it half the roots, so the remainder is the root of the *māl*, we said:
>
> Suppose a *māl* and ten roots *equal* twenty-four dirhams [$x^2 + 10x = 24$]. So we wanted the agreement between the roots and the number. We added to the root the same as the number of half the roots, which is five, since it is appended in this first problem,[34] as we explained before,[35] to get a root and five in number [$x + 5$]. So we multiplied that by itself to get a *māl* and ten roots and twenty-five dirhams [$(x+5)^2 \rightarrow x^2 + 10x + 25$]. And we already had the *māl* and ten roots in the problem *equal* to twenty-four dirhams. So if we put the twenty-four in this multiplication in place of a *māl* and ten roots, then the multiplication of a root and five by itself is forty-nine in number [$(x+5)^2 \rightarrow 49$]. So the root of forty-nine, which is seven, is a root and five. So you subtract from it five, leaving the root to be two, which is the root of the *māl*, and the *māl* is four, and ten roots are twenty. If you added them, it gave twenty-four. So this rule is true analogously in the realm of number.[36]

The proof for type 6 works similarly, except that al-Karajī squares a thing less half the number of roots. Starting with the sample equation $5 + 4x = x^2$ he calculates $(x-2)^2 \rightarrow x^2 + 4 - 4x$. Then he substitutes the x^2 on the right with the $5 + 4x$ from the equation, resulting in $(x-2)^2 \rightarrow 9$. The solution then follows easily.

5. al-Karajī's *al-Kāfī*

Al-Karajī also wrote his *al-Kāfī fī l-ḥisāb* (*Sufficient [book] on calculation*) after *al-Fakhrī*. In it he has a chapter on algebra in which he states and solves the six equations. This time he justifies the rules for solving equations by some of the arithmetical rules that he provides immediately before covering algebra. These rules are not given any proof or explanation. For numbers a, b, c, and s they are:

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[33] [Saidan 1986, 354.6]. As an aside, this passage shows that al-Karajī's *al-Fakhrī* was not intended as a purely scientific work, but was written at least in part as a practical guide for students.

[34] The roots are appended to the *māl* rather than on the opposite side of the equation with the number.

[35] He had explained the completion of the square extensively just before these proofs.

[36] [Saidan 1986, 359.23]. Following this he works it out again for the example $x^2 + 10x = 30$, where the root of the sum is not a perfect square, to get the solution $\sqrt{55} - 5$.

1) $a : b :: ca : cb$.

2) $(a+b)^2 \leftrightarrow a^2 + b^2 + 2ab$.

3) Adding $ab + \left(\frac{1}{2}b\right)^2$ to a^2 makes a square.

4) Subtracting $ab - \left(\frac{1}{2}b\right)^2$ from a^2 makes a square.

5) $ab + |\frac{1}{2}s - a|^2 \leftrightarrow (\frac{1}{2}s)^2$ (where s is the sum $a+b$) (*Elements* II.5).

6) $(s+a)a + \left(\frac{1}{2}s\right)^2 \leftrightarrow (\frac{1}{2}s + a)^2$ (*Elements* II.6).

All of the rules are stated in the text rhetorically. Al-Karajī applies rules 3), 4), and 5) in his proofs, so I give their translations:

> Any square number, if you added to it a number of its roots with the square of half the number of those roots, gave a square.
>
> And if you subtracted from it however many roots you wanted less the square of half the number of those roots, the remainder is a square.
>
> Any number, if you divided it into two different parts and you multiplied one of them by the other, and you added to the outcome the square of the difference between half the number and one of the parts, the outcome is the square of half the number.[37]

Here is the type 4 proof from *al-Kāfī*, which is worked out for the particular equation $x^2 + 10x = 39$. He appeals to rule 3), now adding that the root of the "square" in the rule is "the root of the square and half the number of roots" ($a + \frac{1}{2}b$):

> The first [composite type] is *māl*s and things *equal* a number. For example, a *māl* and ten things *equal* thirty-nine units. So if you wanted to find the unknown, which is the one thing, you halved the number of roots to get five, you squared it to get twenty-five[, then] add it to thirty-nine, since it is *equated* to the one *māl* and ten roots.
>
> And any *māl* and root, if you added to them the square of half the number of these roots, gave a square, its root being equal to the root of the square and half the number of roots. So after that you get sixty-four, and its root is eight. So if you cast away from it the five which is half the roots, it left three, which is a root of the *māl*.[38]

With a as the "thing" and b as 10, rule 3) states that $x^2 + (10x + 25)$ is the square of $x + 5$. Substituting 39 for the $x^2 + 10x$ makes the total 64, and the solution follows. The work takes place in the context of rule 3), and the equation serves to make a substitution, like the proofs in *ʿIlal ḥisāb*.

[37] [al-Karajī 1986, 168.9].

[38] [al-Karajī 1986, 172.7]. After this he covers the normalization of the equation, that is, the setting of the "coefficient" of the *māl*s to 1.

For type 6 the sample equation is "a *māl equal* three roots and four units"[39] ($x^2 = 3x + 4$). Rule 4) is applied, with a as the "thing" and b as 3. The rule states that $x^2 - (3x - 2\frac{1}{4})$ is a square, which this time is $(x - 1\frac{1}{2})^2$. He substitutes the $x^2 - 3x$ with 4, so the total is $6\frac{1}{4}$. The solution follows by equating the roots.

So far the underlying reasoning that al-Karajī follows in *al-Kāfī* is pretty much the same as in *ʿIlal ḥisāb*: he makes a substitution in the rule using the equation. In *ʿIlal ḥisāb* he substitutes the $x^2 + 10x$ with 24 in type 4, the $10x$ with $x^2 + 16$ in type 5, and the x^2 with $5 + 4x$ in type 6. For types 4 and 6 in *al-Kāfī* he substitutes all of the *māl* and roots with the number: $x^2 + 10x$ with 39 in type 4 and $x^2 - 3x$ with 4 in type 6. This way the sum or difference called for in rule 3) or 4) is simply a number. But it is impossible to get a number with such a substitution for type 5. In his sample equation $x^2 + 21 = 10x$ the x^2 is less than $10x$. Instead of looking for some modification of these rules to accommodate this equation type, al-Karajī takes an entirely different approach by using rule 5), an arithmetical version of *Elements* II.5:[40]

> And for the second problem, it is a *māl* and twenty-one units *equal* ten roots. So if you wanted to find the thing, you halved the number of roots and you multiplied it by itself and you subtracted from it the number and you took the root of the outcome, to get two. If you wanted, you added it to half the roots, and if you wanted, you subtracted it from it. This gives the root of the *māl* is truly seven or three.

First he considers the case in which there is one (positive) answer:

> You subtracted the number from the square of half the number of roots because the number is not quite the same as the square of half the number of roots, being either smaller or larger. If it were equal then the number determines it, and it is half the roots of a root of the *māl*, since half of the ten roots which are *equated* to the *māl* and the number is *equal* to the *māl*, and [the other] half is *equal* to the number. This [holds] if the number is equal to the square of half the number of roots.

If the equation is of the form $x^2 + (\frac{1}{2}b)^2 = bx$, then $x = \frac{1}{2}b$, since $\frac{1}{2}bx = x^2$ and $\frac{1}{2}bx = (\frac{1}{2}b)^2$. He expresses these last two as algebraic equations. Next comes the case of two (positive) answers.[41]

> So if [it is] smaller than it,[42] cast away the number, since one portion of the ten roots [say ax] *equals* the *māl*, and the other portion [bx] *equals* the number [21], and when roots *equal* the one *māl* [$ax = x^2$], its number is a root of the *māl* [a is x] as mentioned above.[43] So if you multiplied its number [a] by what is left of the ten that was *equated* with the number [b], then the outcome is equal to the number [21]. So it is clear that the number comes out of what is produced from the multiplication of one part of the number of roots by the other [$a \cdot b \rightarrow 21$].

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[39] [al-Karajī 1986, 176.4].
[40] [al-Karajī 1986, 174.5].
[41] [al-Karajī 1986, 175.2].
[42] I.e. if the number is smaller than the square of half the number of roots.
[43] He wrote earlier how to solve the type $ax = x^2$, which gives $x = a$.

> If the two parts are equal, then half of the number of roots is a root of the *māl*. And if they are different, then the number [21] is always smaller than the square of half the number of roots [25]. If not, then the problem is impossible. So if you cast away the number [21] from it [25], then the root of what remains [2] is the difference between half the number of roots and the part you wanted [$|\frac{1}{2}10 - a|$]. So if you added it to half the number of roots, it gave one of the parts, and if you subtracted it from it, it gave the other part, and either one of them might be the root of the *māl*.

In the equation $10x = x^2 + 21$ the $10x$ is divided into two parts, the x^2 and the 21. Some of these ten things equal the x^2 and the remainder equals the 21. Let's call the parts ax and bx, where $a + b$ is 10. His use of *ʿadala* for "equal" indicates that he is thinking of these as the algebraic equations $ax = x^2$ and $bx = 21$. He solves $ax = x^2$ by the rule for the type 3 equation, giving $x = a$. Then the remainder from the $10x$, which is bx, will equal the 21. Since $x = a$, this is equivalent to the operation $b \cdot a \rightarrow 21$ (not stated as an equation). (We can note that al-Karajī could have obtained this by rewriting the original equation as $21 = 10x - x^2$ and observing that the right side is the product of $10 - x$ by x.) He then appeals to the numerical version of *Elements* II.5. His ab is 21 and $(\frac{1}{2}s)^2$ is 25, so 2, the root of their difference, is $|\frac{1}{2}s - a|$, which in this case is $|5 - a|$. Thus either $5 - a$ or $a - 5$ is 2.

The proofs in *al-Kāfī* are worded rather tersely compared with the proofs in *ʿIlal ḥisāb*, and the proof for type 5 is particularly complex. This suggests that the proofs in *al-Kāfī* were intended for a more sophisticated reader.

6. Ibn al-Yāsamīn's *Talqīḥ al-afkār*

Apart from his time as a student in Seville, Ibn al-Yāsamīn (d. 1204) spent his life in Morroco. Like al-Karajī's *al-Kāfī*, Ibn al-Yāsamīn's textbook *Talqīḥ al-afkār* (*Grafting of Opinions*) is an arithmetic book with a chapter on algebra. Here, too, the six equations and their solutions are explained. Many of Ibn al-Yāsamīn's worked-out problems and his treatment of composite equations are taken from al-Karajī's *al-Fakhrī*. The sections on algebra show no influence of al-Karajī's *al-Kāfī*, but the later section on mensuration has many passages copied from that book.[44]

Like al-Karajī, Ibn al-Yāsamīn gives three rules for solving each type: the standard Arabic rule, Diophantus's rule for solving the non-normalized equation, and Abū Kāmil's rule for finding the *māl* directly. Of these Ibn al-Yāsamīn proves only the standard rule for each composite type in his book. These proofs are merely arithmetical restatements of *Elements* II.5 (for type 5) and II.6 (for types 4 and 6). Here is his rule and proof for type 4:

> If [someone] said to you, a *māl* and ten things *equals* thirty-nine dirhams. How much is the *māl* and how much is the thing?

---

[44] [Abdeljaouad 2005b, 7]. It is possible that someone other than Ibn al-Yāsamīn added these extracts at a later date.

> To work this out you halve the number of roots, which is five, and you multiply it by itself, giving twenty-five. You add it to the thirty-nine to get sixty-four. You take its root, giving eight. You subtract from it half the roots, which is five. The remainder *equals* the one thing, and that is three. And the *māl* comes from multiplying the three by itself, and that is nine.
>
> This is necessarily the way because for every number divided into two halves with an added extension, the multiplication of the number with the extension by the extension and half the original number by itself, is equal to the product of half the number with the extension by itself.[45]

The wording of the proof for the type 5 equation reveals that even if Ibn al-Yāsamīn was writing in the context of arithmetic and algebra, he was still thinking in terms of geometry. In restating *Elements* II.5 he retained the geometrical terms for "longer", "shorter", and "line":

> And the fifth problem: *māl*s and number *equal* roots. Such as when you say: a *māl* and twenty-one dirhams *equals* ten things. So to work it out you halve the things, and that is five. And you multiply it by itself to get twenty-five. You subtract from it the dirhams which are twenty-one dirhams, leaving four. So you take its root, giving two. You subtract it from half the number of roots, leaving three, which is the root of the *māl*, and the *māl* is nine. And if you wanted, you added the two to half the number of roots, to get seven, which is the root and the *māl* is forty-nine.
>
> This is necessary because, for any number divided into two halves and two different parts, the multiplication of the longer (*aṭwal*) part by the shorter (*aqṣar*) part and the multiplication of the excess of half the line (*khaṭṭ*) over the shorter part by itself together are [equal to] that which comes from multiplying half the line by itself.[46]

7. Ibn al-Bannāʾ's algebra book

Our next algebraist also hailed from Morocco. Ibn al-Bannāʾ (1256-1321) wrote his *Kitāb al-uṣūl wa'l-muqaddimāt fī'l-jabr wa'l-muqābala* (*Book on the fundamentals and preliminaries in algebra*), henceforth *Algebra*, in the late thirteenth century. This is a practical guide covering rules of arithmetic, instructions for operating on polynomials, solutions and proofs to the six equations, and 42 worked-out problems. He writes in the beginning of the book:

> I clarified the guiding principles of the work, those for which it is easy to form erroneous impressions and which become clear through understanding. And I did not provide proof[s] (*burhān*) for them, since they are familiar from introductions [to other books], and these introductions rely for the most part on the book of Euclid.[47]

[45] [Ibn al-Yāsamīn 1993, 241.3].
[46] [Ibn al-Yāsamīn 1993, 243.1].
[47] [Saidan 1986, 505.14].

So even if, as we shall see, Ibn al-Bannāʾ gives arguments based in completing the square to justify the rules for solving the composite equations, he does not regard these to be proofs (*barāhīn*) as al-Karajī did. Instead, proofs can be found in other books and are usually grounded in Euclid's *Elements*. Ibn al-Bannāʾ borrowed some worked-out problems from Abū Kāmil's *Book on algebra*, so he may have had that book in mind when wrote of other "introductions".

Like al-Karajī in his *al-Kāfī*, Ibn al-Bannāʾ provides some arithmetical rules, without proof, before addressing the six equations. The section on multiplication alone contains eighteen rules. Only one of these seems to be called on in his proofs for the composite equations: "…multiplying a number by itself…[results in] the two squares of its parts and the surface of one of them by the other twice"[48] ($(a+b)^2 \to a^2 + b^2 + 2ab$). Arithmetical restatements of *Elements* II.5 and II.6 are among the other rules, but they are not used to justify the rules for solving equations.

Ibn al-Bannāʾ gives various rules for solving composite equations, including the standard Arabic rule, a variation on that rule,[49] Diophantus's rule for non-normalized equations, and a rule for finding the *māl* directly for non-normalized equations. The second and fourth of these may be Ibn al-Bannāʾ's own.

For each equation type only the standard rule is proven. Ibn al-Bannāʾ's approach is to complete the square in the context of the equation. Below is his proof for the type 6 equation. In this and in the type 4 and 5 equations this process results in a type 3 equation ($bx = c$), which is easily solved.

> Know that the rule to know the unknown is that you always multiply half the number of assigned roots by itself, and you add the result to the assigned number. So the root of the result is added to half the number of assigned roots. The result then is the root of the sought-after *māl*.
>
> For example, if [someone] said, a *māl equals* three of its roots and four [$x^2 = 3x + 4$]. The rule for this, as described, is that you add the square of half the number of roots, and that is two and a fourth, to the assigned number. Then you add the root of the result to half the number of assigned roots, and the result then is the root of the assigned *māl*, which is four, and the sought-after *māl* is sixteen.
>
> And the rule for this type is summarized in a similar way by means of the fourth and fifth,[50] returning it to the third type, as we made clear, which is that the *māl*, if you subtracted from it the assigned roots, then the remainder *equals* the assigned number. The example becomes: a *māl* less three of its roots *equals* four [$x^2 - 3x = 4$]. And it is clear that if one added to the three roots the same as the square of half the number of roots, the root of the sum is equal to the root of the *māl*, subtracting from it half the number of roots. So one makes the square of

[48] [Saidan 1986, 508.12].
[49] For type 4, for instance, the solution to $x^2 + bx = c$ is, in modern form, $x = \frac{1}{2}(\sqrt{b^2 + 4c} - b)$.
[50] He refers to the type 4 and type 5 equations.

> half the number of roots common,[51] so the equation becomes: a *māl* and two and a fourth less three roots of the *māl equals* six and a fourth [$x^2 + 2\frac{1}{4} - 3x = 6\frac{1}{4}$].
>
> So the root of one side of the equation *equals* the root of the other, and the root of one of the sides is a root of the *māl* less half the number of roots, which is one and a half in this example, and the root of the other is two and a half. So then the equation became: a root of the *māl* less one and a half *equals* two and a half [$x - 1\frac{1}{2} = 2\frac{1}{2}$]. So the root, which is clear from the third type, is four, and the sought-after *māl* is sixteen.[52]

In *ʿIlal ḥisāb* and for the types 4 and 6 equations in *al-Kāfī* al-Karajī worked in the context of the rule for squaring a binomial, using the equation to make a substitution. Ibn al-Bannāʾ, by contrast, works in the context of the equation. For comparison, here are the steps in modern notation of the proofs for the type 6 rule from al-Karajī's *ʿIlal ḥisāb* and Ibn al-Bannāʾ's *Algebra*:

<u>al-Karajī (context of the rule)</u>	<u>Ibn al-Bannāʾ (context of the equation)</u>
$(x-2)^2 \rightarrow x^2 + 4 - 4x$	$x^2 = 3x + 4$
Substitute x^2 with $4x + 5$:	$x^2 - 3x = 4$ Add $2\frac{1}{4}$ to both sides:
$(x-2)^2 \rightarrow (4x + 5) + 4 - 4x$	$x^2 + 2\frac{1}{4} - 3x = 6\frac{1}{4}$ Take the square roots:
$(x-2)^2 \rightarrow 9$ Take the square roots:	$x - 1\frac{1}{2} = 2\frac{1}{2}$
$x - 2$ is 3	x is 4.
x is 5.	

Recall that in *al-Kāfī* al-Karajī had to devise a different way to handle the type 5 equation $x^2 + c = bx$ since subtracting the bx leaves nothing on one side of the equation. Ibn al-Bannāʾ got around this problem for his sample equation $x^2 + 24 = 10x$ by subtacting not the $10x$ from both sides, but $10x - 1$ ("ten roots of the *māl* less one"). This yields the equation $x^2 + 25 - 10x = 1$ ("a *māl* and twenty-five less ten roots of the *māl equals* one"), from which the root of both sides can be taken, and the solutions follow.

The proofs in al-Karajī's *ʿIlal ḥisāb* and in Ibn al-Bannāʾ's *Algebra* are fairly easy to follow. Both books were evidently intended for beginners. And where the proofs in al-Karajī's *al-Kāfī* are decidedly more difficult, those given in this next book of Ibn al-Bannāʾ also require a higher level of mathematical competence.

8. Ibn al-Bannāʾ's *Rafʿ al-ḥijāb*

Ibn al-Bannāʾ wrote his *Talkhīṣ aʿmāl al-ḥisāb* (*Condensed [book] on the operations of arithmetic*) as a brief exposition of calculation with Arabic numerals and methods of finding

[51] To make an amount "common" means to add it to both sides of the equation.
[52] [Saidan 1986, 552.23].

unknown numbers. The chapter on algebra covers only basic rules, including the rules for solving the six simplified equations. No proofs are given in this book.

In 1301 Ibn al-Bannāʾ became the first of over a dozen people to write a commentary on his *Talkhīṣ*. His *Rafʿ al-ḥijāb* (*Lifting the veil*) gives numerical examples, philosophical commentary, and proofs to many of the rules of arithmetic and algebra in the *Talkhīṣ*. In the chapter on algebra he gives two sets of proofs for the rules for solving the three composite equations. In the first set he rests his arguments on a handy rule from finger-reckoning for mentally multiplying numbers, and in the second set his reasoning is founded in arithmetical restatements of *Elements* II.5 and II.6.

First some background on the finger-reckoning rule. Many Arabic arithmetic books describe various tricks for multiplying numbers mentally. Ibn al-Yāsamīn covers several such tricks in his *Talqīḥ al-afkār* that were later copied word-for-word by Ibn al-Bannāʾ into the *Talkhīṣ*. The rule that interests us is described in both books as follows:

> Another method is known as "squaring". You take half of the sum of the two numbers and you square it. You subtract from the result the square of half the difference between them. The remainder is the result of the multiplication.[53]

Neither Ibn al-Yāsamīn nor Ibn al-Bannāʾ give examples of the rule. But al-Hawārī, who wrote a commentary on Ibn al-Bannāʾ's *Condensed Book* in 1305, does:

> For example, we wanted to multiply seventeen by nineteen. So we take half their sum, which is eighteen. We multiply it by itself, which is the meaning of "squaring" as mentioned above, giving three hundred twenty-four. We subtract one, which is the square of half the difference between the two numbers. That leaves three hundred twenty-three, which is the required number, and its figure is 323.[54]

We can write the rule in modern notation as $ab \leftrightarrow (\frac{1}{2}(a+b))^2 - (\frac{1}{2}|b-a|)^2$. Applying it in practice is much simpler than the notational version suggests.

Several rules are described in the chapter on multiplying integers in *Rafʿ al-ḥijāb* that Ibn al-Bannāʾ will later use to justify his two rounds of proofs. He first gives a rephrasing of the "squaring" rule, together with a variation as well as another rule. These are formulated in precisely the form in which he will use them in the proofs:

> The first type of the types of multiplication, known as "squaring", deems it necessary that the surface of two numbers with the square of half the difference between them is equal to the square of half their sum. And the product of one of them by the other subtracted from the square of half their sum leaves the square of half the difference between them, and the square

[53] [Ibn al-Yāsamīn 1993, 116; Ibn al-Bannāʾ 1969, 51.7].
[54] [al-Hawārī 2013, 113.1].

of half the difference between them is clearly the square of the difference between one of them and half their sum.[55]

In modern notation these three rules are:

1) $ab + (\frac{1}{2}|b - a|)^2 \leftrightarrow (\frac{1}{2}(a + b))^2$,
2) $(\frac{1}{2}(a + b))^2 - ab \leftrightarrow (\frac{1}{2}|b - a|)^2$
3) $(\frac{1}{2}|b - a|)^2 \leftrightarrow |\frac{1}{2}s - a|^2$, where s is $a + b$.

Next Ibn al-Bannāʾ restates rule 1) three ways, equating each term with the sum or difference of the other two. I number these rules 4), 5), and 6), but since he does not appeal to them, I will not translate them. He then gives his arithmetical restatements of *Elements* II.5 and II.6, neither of which is found in the *Talkhīṣ*. These will be called on in the second round of proofs:

> Another rule is that for every number divided into two halves and into two different parts, the product of one of the different parts by the other with the square of the difference between one of them and half the number is equal to the product of half the number by itself, since you set the two different parts as the two multipliers.
>
> Another rule is that for every number divided into two halves with another number added to it, the product of the number with the added part by the added part and the square of half the number is equal to the product of half the number and the added part by itself, since you set the number with the added part as one of the multipliers and you set the added part as the other multiplier. So the difference between them is the number divided into two halves.[56]

The first of these is adapted from *Elements* II.5, and the second from *Elements* II.6. Contorted into modern notation they are:

7) $ab + |a - \frac{1}{2}s|^2 \leftrightarrow (\frac{1}{2}s)^2$, where s is $a + b$.
8) $(s + a)a + (\frac{1}{2}s)^2 \leftrightarrow (\frac{1}{2}s + a)^2$, where a is an extension added to s.

In his first round of proofs Ibn al-Bannāʾ will use rule 1) to prove types 4 and 6, first calculating the two terms on the left. Rule 2) is merely a rearrangement of the terms of 1), and he uses it in this form, as a difference, for his type 5 proof. That proof also requires that he make the substitution called for in rule 3), which has the effect of turning the finger-reckoning rule into the arithmetical version of *Elements* II.5.

By making the two numbers a and b the *māl* and the number, rules 1) and 2) allow Ibn al-Bannāʾ to reduce the original equations to type 1 equations ($ax^2 = bx$). If that isn't simple enough, he further adjusts them to type 3 equations ($bx = c$). He performs his proofs tersely,

[55] [Ibn al-Bannāʾ 1994, 260.5].
[56] [Ibn al-Bannāʾ 1994, 260.16].

without reference to a specific equation. Here is the type 4 proof, interspersed with my explanations in terms of the specific equation $x^2 + 10x = 39$:[57]

> And the cause (ʿ*illa*) of the procedures for the composite types you can understand from the multiplication by "squares" that we mentioned before. You always make the two multiplied numbers the number and the *māl*.

In rule 1) the a and b are 39 and x^2.

> Then the difference between them is the things in the fourth type.

In our type 4 equation their difference is $39 - x^2$, which are "things": $10x$.

> So you multiply one of them by the other giving *māl*s [$39x^2$], and you add to it the square of half their difference, giving *māl*s [$39x^2 + 25x^2 \rightarrow 64x^2$]. So this is the square of half their sum. You take its root, giving half their sum, which are things [$8x$]. So keep it in mind.

Now $ab + (\frac{1}{2}|b - a|)^2$ has been calculated as $64x^2$, whose square root is $8x$. This must be equal to $\frac{1}{2}(a + b)$:

> Then you direct your attention to half their sum to get a *māl* and half the things which are with it. Since the number is equal to the *māl* and the things, everything with the *māl* are two *māl*s and the things, and half of that is a *māl* and half the things.

Adding an x^2 to both sides of $x^2 + 10x = 39$ gives $2x^2 + 10x = 39 + x^2$, which is their sum. So half the sum is half the left side, which is $x^2 + 5x$.

> Confront it with the remembered amount, leaving things *equal* a *māl*, which is the first type.

A common way to express the creation of an equation from two algebraic expressions was to "confront" (from *qabila*) them. By this act the equation $x^2 + 5x = 8x$ is established, which simplifies to a type 1 equation.

> And if you wanted, you added half the things to the remembered amount, to get things *equal* the number, and that is the third type.

Adding $5x$ to both sides of $x^2 + 5x = 8x$ gives $x^2 + 10x = 8x + 5x$, which becomes $39 = 8x + 5x$ by a substitution from the equation. This simplifies to the type 3 equation $39 = 13x$.

Ibn al-Bannāʾ next proves the rule for type 6, which works similarly. The type 5 equation is solved a little differently. I will explain the procedure in terms of the specific equation $x^2 +$

[57] [Ibn al-Bannāʾ 1994, 309.14]. This proof is explained again in terms of our equation in [Ibn al-Hāʾim 2003, 83.2].

$21 = 10x$. Now the difference between the number and the *māl* is not the things, since the two are on the same side of the equation.[58]

> And in the fifth type the square of half the things is the square of half the sum of the numbers.

In $x^2 + 21 = 10x$ the square of half of $10x$ is the square of $\frac{1}{2}(x^2 + 21)$. This is $25x^2$.

> So you subtract from it the multiplication of one of them by the other [$25x^2 - 21x^2 \rightarrow 4x^2$], leaving the square of the difference between one of them and half their sum.

By rule 2) this subtraction leaves $(\frac{1}{2}|b - a|)^2$, but this is equal to $|\frac{1}{2}s - a|^2$ by rule 3). So $4x^2$ is $|\frac{1}{2}s - a|^2$. Because either a or b can be the 21, $4x^2$ is either $|5x - x^2|^2$ or $|5x - 21|^2$. And because we do not know which of $\frac{1}{2}s$ ($5x$) or a (21 or x^2) is larger, the root of $4x^2$ could be $x^2 - 5x$, $21 - 5x$, $5x - x^2$, or $5x - 21$. He considers the four cases in order:

> So if you took its root to get things [$2x$], and you added it to half their sum, which is half the things which are in the equation [$5x$] and you *equated* that with the *māl*, it resulted in the first type [$7x = x^2$], and if you *equated* it with the number it gave the third type [$7x = 21$].
>
> And if you subtracted [from the $5x$] the root of half their remembered sum [$2x$] and you *equated* it with the *māl*, it likewise resulted in the first type [$3x = x^2$], and if you *equated* it with the number it resulted in the third type [$3x = 21$].

The proofs in the second round are structured like those in the first. The arithmetical rules on which the arguments rest, now taken from *Elements* II.5 or II.6, again allow Ibn al-Bannāʾ to transform his equations into type 3 and type 1 equations. Here, too, the proofs unfold in general terms, without reference to a sample equation. Here is his type 6 proof, with my explanations in terms of the specific equation $x^2 = 4x + 5$:[59]

> And in the sixth type you divide the things into two halves and you make the number the added part.

In the context of rule 8), the arithmetical version of *Elements* II.6, s is $4x$ and a is 5.

> So the product of the whole, which is the *māl equated* to them, by the added part, with the square of half the things, is equal to the product of half the things and the number by itself, as quantities.

The "whole" is the two terms $4x$ and 5 together, which by the equation equals x^2. By rule 8), $x^2 \cdot 5 + (2x)^2$ is equal to $(2x + 5)^2$. This is not expressed as an algebraic equation, but is a restatement of the arithmetical rule using the algebraic terms.

[58] [Ibn al-Bannāʾ 1994, 310.7].
[59] [Ibn al-Bannāʾ 1994, 311.1]. This proof is explained again in terms of our equation in [Ibn al-Hāʾim 2003, 97.1].

> The root of that, which is things, is equal to half the things with the number.

The left part adds to $9x^2$, so taking the root of both parts we get that $3x$ must equal the $2x + 5$. This will now be expressed an equation:

> So you confront them, resulting in the third type,

The confrontation gives the equation $3x = 2x + 5$, which simplifies to a type 3 equation.

> or you add half the things to the root and you confront that with the *māl*, resulting in the first type.

Adding $2x$ to the $2x + 5$, the root of $(2x + 5)^2$, gives $4x + 5$, which by the original equation is x^2. One must also add $2x$ to the other part, $3x$, yielding the equation $5x = x^2$, which is a type 1 equation.

The type 4 equation is proven similarly, but the added part is the *māl* instead of the number. For type 5 he appeals to rule 7), the arithmetical version of *Elements* II.5. In the equation $x^2 + 21 = 10x$, for instance, s is the $10x$, which is divided unequally into the x^2 and the 21.

9. Ibn al-Hāʾim's commentary on Ibn al-Yāsamīn's poem

In addition to his treatment of algebra in *Talqīḥ al-afkār* Ibn al-Yāsamīn composed a brief 54-line *al-Urjūza fī'l-jabr wa'l-muqābala* (*Poem on algebra*), in which he gives the basic rules of algebra including concise instructions for solving the six equations.[60] Like Ibn al-Bannāʾ's *Condensed book*, Ibn al-Yāsamīn's poem inspired numerous commentaries by later mathematicians. One of these, Ibn al-Hāʾim's *Sharh al-urjūza al-Yāsmīnīyya* (*Commentary on the Poem of al-Yāsamīn*), provides an assortment of arithmetical proofs.

Shihāb al-Dīn Ibn al-Hāʾim was born in Cairo in 1352. He taught mathematics in Jerusalem and he wrote his commentary during a pilgrimage to Mecca in 1387. He cites several previous books, including the algebra books of al-Khwārizmī and Abū Kāmil, al-Karajī's *al-Fakhrī* and *al-Badī*ʿ, and Ibn al-Bannāʾ's *Algebra* and *Talkhīṣ*.[61] He was also familiar with Ibn al-Bannāʾ's *Rafʿ al-ḥijāb*, since he paraphrases proofs from that book.

In his chapter "On solving the six problems" Ibn al-Hāʾim gives numerous rules and proofs borrowed from various books. He gives four proofs for the type 4 equation, two for type 5, and four for type 6. His proofs are nearly all adapted from al-Karajī's *al-Fakhrī* and Ibn al-Bannāʾ's *Algebra* and *Rafʿ al-ḥijāb*. A breakdown is given in the Appendix.

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[60] The poem is edited and translated into English in [Abdeljaouad 2005].
[61] [Ibn al-Hāʾim 2003, 9].

Just before he begins his proofs he makes a remark about geometric proofs to the rules for solving the composite equations:

> It is common for people to explain the proofs (*barāhīn*) of these problems [i.e. equations] by geometry, using lines or surfaces. Since understanding them requires knowledge of Euclid, I thought to show numerical preliminaries that do not rely on line or surface, even if these preliminaries themselves have need of geometric proofs (*barāhīn*). I do this to make the understanding easier, relegating the demonstration[s] (*bayān*) of these preliminaries to Euclid or to some other book on geometry.[62]

The "numerical preliminaries" on which Ibn al-Hāʾim's wide variety of proofs rests include every rule we have seen so far: completing the square, arithmetical versions of *Elements* II.5 and II.6, and the finger-reckoning rule used by Ibn al-Bannāʾ. And what we might have inferred from the similar remark in Ibn al-Bannāʾ's *Algebra* is made explicit here: these arithmetical preliminaries should be proven by geometry. Geometric proofs are bypassed in favor of these preliminaries to make the reasoning easier for arithmetic students, just as we first saw in al-Karajī's *ʿIlal ḥisāb*.

I will not run through all the different ways that Ibn al-Hāʾim proved the rules for solving the three composite equations, since we have already reviewed them in the books from which he borrowed. The only proofs that might be of interest are the ones he adapted from the geometric proofs in al-Karajī's *al-Fakhrī*. Below is his first proof for the type 4 equation, which begins with his arithmetical version of *Elements* II.6.

> I say: for any number divided into two halves with another number added to it, what is obtained from multiplying the number with the added part by the added part, and then adding to it the square of half the number, gives a result equal to the multiplication of all of the added part and half the number by itself…[63]
>
> This being established, we consider the expression in the first example, which is a *māl* and ten roots *equals* twenty-four [$x^2 + 10x = 24$]. So we say that the number of roots [10] is the main number, and the number of the roots of the *māl* [i.e. the value of x] paired with it is the number added to it. And the simple number [24] is the outcome of the multiplication of the number with the added part by the added part [$(10 + x) \cdot x \rightarrow 24$ *from the equation*]. So the twenty-four in the example comes from the multiplication of the ten and the number of the roots of the *māl* added to it by the number of added roots.
>
> So we halved the number of roots, and we squared that half, and we added the outcome, which is twenty-five, to the number. The sum is forty-nine, which is equal to the square of the sum of the number of the roots added to the ten [x] and half of the ten. So the root of forty-nine, which is seven, is all of half the number of roots [5] and the number of the roots added to the ten [x]. So if one subtracted from the seven half the ten, it left two, which is the number of the roots of the *māl* added to the ten roots…[64]

[62] [Ibn al-Hāʾim 2003, 79.17].
[63] [Ibn al-Hāʾim 2003, 79.22].
[64] [Ibn al-Hāʾim 2003, 80.5].

Like al-Karajī, Ibn al-Hāʾim makes the number the number of roots (10) and the added part a root (x), and all work takes place in the context of the rule. Compare this with Ibn al-Bannāʾ's procedure in the second round of proofs in *Rafʿ al-ḥijāb* for the solution to $x^2 + bx = c$. He also appealed to a numerical version of *Elements* II.6, but in his proof the "number" was all of the bx, and the added part was the whole *māl* (x^2). Also, Ibn al-Bannāʾ used the rule to set up type 1 and type 3 equations in the end, while Ibn al-Hāʾim works entirely within the context of the arithmetical version of *Elements* II.6.

10. al-Fārisī's *Asās al-qawāʿid*

Kamāl al-Dīn al-Fārisī (d. ca. 1320) was a Persian mathematician well known today for his work in optics and number theory. His book *Asās al-qawāʿid* (*Foundation of Rules*) is a commentary on a guide to practical mathematics by ʿImād al-Dīn ibn al-Khawwām al-Baghdādī (1245-1325). The two books cover basic arithmetic, mensuration, algebra, and double false position, with 44 worked-out problems that in al-Fārisī's commentary are solved by various methods.

One of al-Fārisī's stated goals is to provide proofs to the rules given throughout al-Baghdādī's book. Al-Fārisī's view of what constitutes a proper proof for arithmetical rules differs from the views of Ibn al-Bannāʾ and Ibn al-Hāʾim. Instead of holding that geometry is a (or the) proper domain of proof, he maintains that propositions in arithmetic (and thus also algebra) should be performed in the domain of arithmetic. He expresses this view just after giving his proofs to the rules for solving the composite equations:

> I say: proofs (*barāhīn*) in arithmetic, rather, are achieved by means of arithmetic. In any problem, you relate types with types, [that is,] with what it is associated with it of that type. However, when they[65] saw the common relationship between sides of the surfaces and the surfaces as the [geometric] lines and surfaces, and as the numerical sides and surfaces, they established in many instances proof (*dalīl*) of the arithmetical by means of lines and surfaces and bodies. But it is more proper whenever possible to firmly established [them] in number, so they should not be demonstrated by means of lines.[66]

Many of the worked-out problems in al-Baghdādī's book are taken from al-Karajī's *al-Fakhrī*, but it is unclear if al-Fārisī was familiar with this source. Al-Fārisī did know al-Karajī's *al-Kāfī*, however, since he cites that book in his chapter on mensuration.[67] Later, in his chapter on algebra, al-Fārisī rests his proofs for solving composite equations on arithmetical preliminaries. But unlike other algebraists, he gives arithmetical proofs to these preliminaries. His arguments follow the reasoning in al-Karajī's *al-Kāfī*. Below is his treatment of the type 4 equation. He first

[65] I suppose by "they" he means earlier algebraists.
[66] [al-Fārisī 1994, 524.13].
[67] Al-Baghdādī's first nine problems are taken, sometimes with minor modifications, in order from problems I.1-I.9 in *al-Fakhrī*. Also, al-Fārisī's problems (10) through (14) are al-Karajī's problems I.13, I.15, I.16, and I.18. Al-Fārisī mentions al-Karajī's *al-Kāfī* starting at [al-Fārisī 1994, 421.11].

gives the rule from al-Baghdādī's book, with the solution for the specific example $x^2 + 10x = 39$.[68]

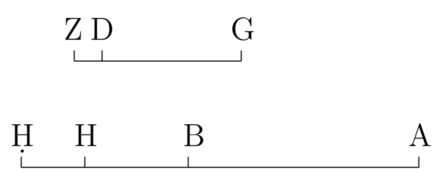
> [al-Baghdādī] said: Then you square half the number of things and you add it to the number, and you take the root of the outcome, and you cast away from it half the things, so what is left is the thing.
>
> For example: a *māl* and ten things *equal* thirty-nine dirhams [$x^2 + 10x = 39$]. The square of half the number of things is twenty-five. You add it to the number to get sixty-four. Its root is eight, and you cast away from it half the roots, leaving three, which is the thing, and the *māl* is nine.

Al-Fārisī then states the arithmetical identity $(a + b)^2 \leftrightarrow a^2 + b^2 + 2ab$ and the rule for completing the square which follows from it. This second rule is stated in arithmetical terms with the word "square" (*murabbaʿ*) and not the algebraic *māl*. And where the first rule is about a number divided into two parts, the second involves the act of adding a quantity to complete the square:

> I say: let us give two introductory remarks. The first is that for any two different numbers, the square of the larger is equal to the squares of the smaller and the difference between them, and double the surface of the smaller by the difference. And that is because the larger is divided into the two parts of the smaller and the difference, and this statement is clear, having been mentioned several times before.[69]
>
> And the second is that if one added to a square and a number of its roots the square of half its number, then the outcome is a square. Its root is the root of the original square with half the number.

Al-Fārisī then gives an arithmetical proof of the first rule.

> Let AB be the square of GD, with BH added to it, which is the number of its roots. Let half the number be DZ, and its square HḤ. So I say: AḤ is the square of GZ, and that is because the square of GZ is equal to the two squares of GD, DZ and double the surface GD by DZ. And AB is the square of GD, and HḤ is the square of DZ, and double GD by DZ is BH since it is equal to GD by double DZ. So AḤ is the square of GZ and that is what we wanted.



[68] [al-Fārisī 1994, 519.20].

[69] He had used this fact, also stated originally as a difference, in a proof in the chapter on subtracting roots [al-Fārisī 1994, 496.15]. I could not find it anywhere else in his book.

The rule for completing the square is then invoked to explain the rule for solving the general type 4 equation:

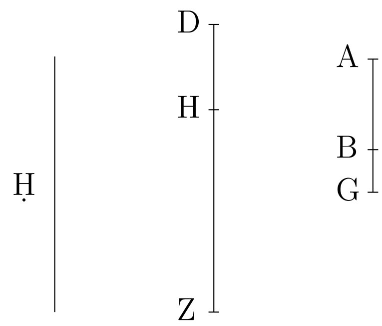
> And then I say: if a *māl* and things *equal* a number, and to that number is added the square of half the number of things, then the outcome is the square of the number [obtained from] adding half the number of things to the root of the *māl*. So if [we] find its root and subtract from it half the number of things, then the remainder is the root of the *māl* which was sought.

Al-Fārisī's treatment of the type 6 equation is similar. Recall that for type 5 al-Karajī resorted to a different kind of argument, perhaps because a simple completing of the square cannot work. Al-Fārisī follows al-Karajī's proof, which he prefaces with four lemmas:[70]

> [al-Baghdādī] said: the fifth problem, which is the second of the connected [problems]: *māl*s and number *equal* things.
>
> And the way to find the thing, after returning and completing, is that you square half the number of roots and you cast away the number from this, and you take the root of the remainder. If you wanted, add it to half the roots, and if you wanted, subtract it from it. What remains after the addition or subtraction is the thing.
>
> For example, a *māl* and twenty-one dirhams *equal* ten things. The square of half the number of things is twenty-five. We cast away the number from it, leaving four. The root of that is two. If you wanted, you added to it five, and if you wanted, you subtracted it from it. So the thing, if you wanted, is seven, and if you wanted, is three.
>
> And if the number is larger than the square of half the number of things, then the problem is impossible. And if [it] is equal to it then half the number of things is the thing.
>
> I say: it is clear after [these] premises:
>
> First, for any two numbers, the sum of their squares is equal to double the surface of one of them by the other if they are equal (if a and b are equal, then $a^2 + b^2 \leftrightarrow 2ab$), and adding to double the surface the square of their difference if they are different (if $a \neq b$ then $a^2 + b^2 \leftrightarrow 2ab + |a - b|^2$).
>
> The first it is clear, and for the second, let the two numbers be AB, AG, and their squares be DH, HZ, and double their surface be Ḥ. Now the square of AG[71], namely HZ, is equal to the two squares of AB, BG and double the surface AB by BG. So DH, HZ, namely DZ, is equal to the square of BG and double square of AB and double AB by BG, and surface AB by AG is equal to the square of AB and surface AB by BG. So double AB by AG, namely Ḥ, is equal to double the square of AB and double AB by BG. So DZ exceeds Ḥ by the square of BG, and that is what we wanted.

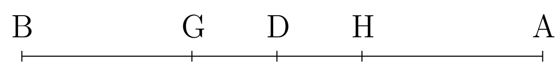
[70] [al-Fārisī 1994, 520.16].

[71] Text says AB.



Second: For any number divided unequally, the difference between its half and either of them is half the difference between them.

Let the number AB be divided into the two numbers AG, GB differently, and let AD be half of AB, and subtract BG from AD, leaving HD. Now BG subtracted from two equal numbers BD, AD, leaves remainders, namely GD, DH, likewise equal. And HG is the difference between AG, BG, namely AH. And the difference between AD and each of AG, BG is half of GH, and that is what we wanted.



Third: If a *māl* and number *equal* things, then a portion of those things are *equal* to the *māl*, and let its number be A, and the remainder are *equal* to the number, and let its number be B, and [let] the number of things be G. Clearly A by the thing is the *māl*, so A is the thing, and B by the thing is the number. So I say, the square of half of G either equals the number which is with the *māl* or it exceeds it.

In an equation of the form $x^2 + c = bx$ the bx is divided into two parts that I can call αx and βx, which are equal to the x^2 and the c respectively. His A is my α, and his B is my β. I use different notation to avoid the anachronistic forms Ax and Bx. "A by the thing is the *māl*" because $\alpha \cdot x$ is x^2. So A (which is α) is "the thing" x. Likewise, "B by the thing is the number", or $\beta \cdot x$ is c.

If it is smaller then it is impossible, since half of G is half of A with half of B, and its square is equal to the two squares of half of A and half of B, and double the surface of half of A by half of B, namely the surface A by half of B. And the square[s] of half of A and half of B together exceed double their surface, namely surface A by half of B, by the square of their difference if they are unequal, and they are equal if they are equal.[72] But surface A by half of B is half of the number which is with the *māl*. Since surface A by B is the number, the square of half of A and half of B with half the number, namely, the square of half of G, exceeds the number by the square of the difference or it is equal to it, and that is what we wanted.

[72] By the first rule.

> Fourth: If the square of half of G equals the number, then the halves of A, B are equal. Otherwise suppose they are different. If one follows this, then the square of half of G exceeds the number, and this is a contradiction. And if it is greater than it, it must be that the amount of the excess is the square of the difference between the two halves [of A and B], since the two halves will not be equal.[73] In fact it is necessary that [they be] different, so the square of their sum, namely half of G, is greater than the number by the square of the difference, and that is what we wanted.
>
> And then we say, if the square of half the number of things is equal to the number, then half the number of things is the thing, since the two halves A, B are equal.[74] So A, B are equal, and A is the thing so B is likewise. So the number is equal to the *māl*, and its root—namely half the number of things—is the thing.
>
> If the square of half the number [of things] is greater than number, you took a root of the excess, which is the difference between the two halves of A, B.[75] And its double is the difference between A, B. Now half the difference between A, B, namely the root of the excess, is the difference between half of all of A, B, namely half of G, and between each of A, B.[76] So if you subtracted it from half of G, namely half of the number of things, one of them remains [i.e. A or B], and if you added it to it, the outcome is the other. You can make whichever you want of the remainder or the outcome the thing. Either way you get the answer to the two sides as mentioned, and that is the answer.

Al-Fārisī's *Asās al-qawāʿid* is not a book for beginners. The proofs he gives throughout the book to basic results in arithmetic and algebra are written at a level beyond the abilities of students learning the material for the first time. This work, like al-Karajī's *al-Kāfī* and especially Ibn al-Bannāʾ's *Rafʿ al-ḥijāb*, considers elementary techniques from a more theoretical perspective.

11. Concluding remarks

The earliest geometric proofs for the rules for solving equations do not rely on Euclid. The arguments of al-Khwārizmī and Ibn Turk appeal to an intuitive manipulation of lines and areas that would have been familiar to those who worked in mensuration. Later in the century Abū Kāmil introduced Euclid to practical algebra, and the arguments now presume knowledge of foundational propositions in Books I and II of the *Elements*. It is not surprising that the proofs in al-Karajī's *al-Fakhrī*, which are similarly based in Euclid and use single-line diagrams, were difficult for arithmetic students to grasp. By shifting proofs to a foundation in Greek geometry these algebraists removed them from the grasps of many arithmetic students studying algebra.

To rectify the problem al-Karajī could have returned to geometric proofs in the style of al-Khwārizmī. Instead, he took a different direction in his *ʿIlal ḥisāb* by giving arithmetical proofs based on derivations he had learned from Diophantus's *Arithmetica*. These proofs are indeed

[73] By the third rule.
[74] By the first part of the fourth rule.
[75] By the second part of the fourth rule.
[76] By the second rule.

easier to understand and require no knowledge of Euclid. But ease of understanding could not have been the motive for the arithmetical proofs he gives in his *al-Kāfī*. There the proofs are phrased in a terse manner and the type 5 proof is complex, so they would not have been accessible beginners.

We see the same two approaches in the books of Ibn al-Bannāʾ, written over two and a half centuries later in Morocco. His book on algebra gives easily understandable proofs based on completing the square. But the two sets of proofs in his *Rafʿ al-ḥijāb* are as clever as they are difficult. Not only are they expressed as tersely as the proofs in al-Karajī's *al-Kāfī*, but they are worked through without reference to sample equations. Both al-Karajī and Ibn al-Bannāʾ seem to have taken the challenge of writing arithmetical proofs as a theoretical excercize in these second books.

The proofs of al-Karajī and Ibn al-Bannāʾ were later copied in different ways by the other three algebraists. Ibn al-Yāsamīn apparently based his short treatment on the geometric proofs in al-Karajī's *al-Fakhrī*. His "proofs" consist only of arithmetical versions of the statements of Euclid's *Elements* Propositions II.5 and II.6. For his part Ibn al-Hāʾim simply collected together proofs from the books of al-Karajī and Ibn al-Bannāʾ. The arguments for al-Fārisī's proofs are based on those in al-Karajī's *al-Kāfī*. His contribution lies in insisting that the arithmetical preliminaries receive numerical proofs.

What constitutes a proof for these algebraists? Al-Karajī writes that just as his geometric proofs from *al-Fakhrī* are sound, so are his arithmetical proofs in *ʿIlal ḥisāb*. The arithmetical proofs in the latter work stand on their own, and have no need of a geometric foundation. A proof for him seems to be any argument founded on known results, whether in Greek geometry or in accepted arithmetical rules. Ibn al-Bannāʾ, instead, does not believe that arguments based in completing the square can be considered proofs, and he refers to geometric proofs of these preliminaries in other books. Ibn al-Hāʾim agrees, saying that the arithmetical preliminaries should receive geometric proof. We find still another view in al-Fārisī, who maintains that propositions in arithmetic (and thus also in algebra) should be proven arithmetically, and not by geometry. He gives proofs in the style of Euclid's number theory books for the same kinds of arithmetical rules that Ibn al-Bannāʾ and Ibn al-Hāʾim saw as requiring geometric proof.

All this leads me to reconsider what I wrote about the proofs in Abū Kāmil's *Algebra* in [Oaks 2011]. There I noted how Abū Kāmil proved rules given in terms of specific examples by geometry, and proofs of results stated in general terms by arithmetic (in the style of Euclid's number theory books). I suspected there was a dilemma due to the incompatibility of the numbers of Arabic arithmeticians with both Euclid's numbers and Euclid's geometric magnitudes. Arabic numbers include fractions and irrational roots, while Euclid's numbers are restricted to integers. And where Arabic numbers are homogeneous, Euclid's magnitudes possess dimension. Because of this dual incompatibility, neither form of proof in Abū Kāmil is truly grounded in Euclid's *Elements*.

The apparent lack of concern over this issue in the authors reviewed in this article suggests that my perceived dilemma may not have been seen as a problem for Abū Kāmil. Perhaps he simply followed al-Khwārizmī's style for propositions stated in terms of specific examples, and he followed Euclid's style for propositions stated in general terms. Arabic algebraists seem to have been willing to borrow only part of the axiomatic-deductive structure of Greek geometry, namely the style of proof, without adopting an underlying system of postulates. This makes the question of what constitutes a proof a subjective one, and indeed we see disagreement among our algebraists.

We knew before that the presence of proofs in a book is not necessarily an indication of a Greek axiomatic-deductive framework. Abū'l-Wafāʾ noted that practical geometers wrote proofs to rules of constructions that were unrelated to Greek geometry, and the proofs in al-Khwārizmī and Ibn Turk agree with their cut-and-paste style.[77] To these proofs we can now add the arithmetical proofs of al-Karajī (*ʿIlal ḥisāb*) and Ibn al-Bannāʾ (*Algebra*) which were designed explicitly for practitioners. And it should not be surprising that these mathematicians took the idea a step further to explore arithmetical proofs on a more sophisticated level in their books *al-Kāfī* and *Rafʿ al-ḥijāb*, and that these proofs remain disconnected with Greek mathematics.

Appendix. Summary of the methods of proof in the five authors.

Foundations: (1) Completing the square
(2) Arithmetical versions of *Elements* II.5 for type 5: $ab + \left|b - \frac{1}{2}s\right|^2 \leftrightarrow \left(\frac{1}{2}s\right)^2$, and II.6 for types 4 and 6: $(s+a)a + (\frac{1}{2}s)^2 \leftrightarrow (\frac{1}{2}s + a)^2$.
(3) The rule from finger-reckoning: $ab + (\frac{1}{2}|b-a|)^2 \leftrightarrow (\frac{1}{2}(a+b))^2$

Numbering: For example, 5b is the second proof of the author for the type 5 equation.

al-Karajī's *al-Fakhrī*

4a	$x^2 + 10x = 39$	(2) by geometry with $s = 10$ and $a = x$
4b		(1) derivation from Diophantus
5a	$x^2 + 21 = 10x$	(2) by geometry with $s = 10$ and $a = x$
5b		(1) derivation from Diophantus
6a	$x^2 = 3x + 4$	(2) by geometry with $s = x$ and $a = 3$.

"Method of Diophantus" performed in the context of the operation of squaring the thing and/less half the roots.

al-Karajī's *ʿIlal ḥisāb* (*Causes of calculation*)

4c	$x^2 + 10x = 24$	(1) square $x + 5$, make substitution.
5c	$x^2 + 16 = 10x$	(1) square $x - 5$ or $5 - x$, make substitution.
6c	$5 + 4x = x^2$	(1) square $x - 2$, make substitution.

The "rule of Diophantus" is converted into proofs for the standard case.

[77] [Høyrup 1986, 473].

al-Karajī's *al-Kāfī*

4d	$x^2 + 10x = 39$	(1) work with $(x+5)^2 \rightarrow 64$.
5d	$x^2 + 21 = 10x$	(2) $s = 10$, $a = x$.
6d	$x^2 = 3x + 4$	(1) work with $(x - 1\frac{1}{2})^2 \rightarrow 6\frac{1}{4}$.

Ibn al-Yāsamīn's *Talqīḥ al-afkār* (*Grafting of opinions*)

4	$x^2 + 10x = 39$	(2) *Elements* II.6 is restated arithmetically
5	$x^2 + 21 = 10x$	(2) *Elements* II.5 restated arithmetically, w/geometric terms
6	$x^2 = 3x + 4$	(2) *Elements* II.6 is restated arithmetically

Ibn al-Bannāʾ's *Algebra*

4a	$x^2 + 10x = 39$	(1) Add 25 to both sides.
5a	$x^2 + 24 = 10x$	(1) Subtract $10x - 1$ from both sides.
6a	$x^2 = 3x + 4$	(1) From $x^2 - 3x = 4$ add $2\frac{1}{4}$ to both sides.

All are worked out in the context of the equation.

Ibn al-Bannāʾ's *Rafʿ al-ḥijāb* (*Lifting the Veil*)

4b	$x^2 + cx = d$	(3)
6b	$x^2 = cx + d$	(3) Same procedure, which is why equation 6 is placed before 5.
5b	$x^2 + d = cx$	(3) Uses $\left(\frac{1}{2}\lvert b - a\rvert\right)^2 = \lvert b - \frac{1}{2}(a+b)\rvert^2$ to reduce it to *Elements* II.5. Depending which of a, b is x^2 and d gives the two different solutions.

In each proof a and b are x^2 and d. By substitutions simple equations of types 1 and 3 can be found ($ax^2 = bx$ or $ax = b$).

4c	$x^2 + cx = d$	(2) For $(s+a)a + (\frac{1}{2}s)^2 = (\frac{1}{2}s + a)^2$, $s = cx$ and $a = x^2$.
5c	$x^2 + d = cx$	(2) For $ab + \lvert b - \frac{1}{2}s\rvert^2 = (\frac{1}{2}s)^2$, $s = cx$ and a, b are x^2, d. Depending which is which gives the two different solutions.
6c	$x^2 = cx + d$	(2) For $(s+a)a + (\frac{1}{2}s)^2 = (\frac{1}{2}s + a)^2$, $s = cx$ and $a = d$.

For each equation substitutions are made to get simple equations of types 1 and 3 ($ax^2 = bx$ or $ax = b$).

Ibn Hāʾim's commentary on al-Yāsamīn's poem

4a	$x^2 + 10x = 24$	(2) For $(s+a)a + (\frac{1}{2}s)^2 = (\frac{1}{2}s + a)^2$, $s = 10$ and $a = x$. Context of the rule. Adapted from al-Karajī's 4a.
4b	$x^2 + 10x = 39$	(3) Follows Ibn al-Bannāʾ's 4b.
4c	$x^2 + 10x = 39$	(2) For $(s+a)a + (\frac{1}{2}s)^2 = (\frac{1}{2}s + a)^2$, $s = 10x$ and $a = x^2$. Follows Ibn al-Bannāʾ's 4c.
4d	$x^2 + 10x = 39$	(1) Follows Ibn al-Bannāʾ's 4a.
5a	$x^2 + 16 = 10x$	(2) $ab + \lvert b - \frac{1}{2}s\rvert^2 = (\frac{1}{2}s)^2$, $s = 10$ and $a = x$. Follows al-Karajī's 5a.
5b	$x^2 + 16 = 10x$	(2) For $ab + \lvert b - \frac{1}{2}s\rvert^2 = (\frac{1}{2}s)^2$, $s = 10x$ and a, b are x^2, 16.

Follows Ibn al-Bannāʾ's 5c.

6a	$x^2 = 10x + 24$	(1) Source unknown.
6b	$x^2 = 10x + 24$	(3) Follows Ibn al-Bannāʾ's 6b.
6c	$x^2 = 10x + 24$	(2) Follows Ibn al-Bannāʾ's 6c.
6d	$x^2 = 10x + 24$	(1) Follows Ibn al-Bannāʾ's 6a.

al-Fārisī's *Asās al-qawāʿid* (*Foundation of rules*)

4	$x^2 + cx = d$	(1)
5	$x^2 = cx + d$	(2)
6	$x^2 + d = cx$	(1)

All proofs are elaborations on the proofs in al-Karajī's *al-Kāfī*.

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